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Relaxing the conditions of ISS for multistable periodic systems

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Abstract The input-to-state stability property of nonlinear dynamical systems with multiple invariant solutions is analyzed under the assumption that the system equations are periodic with respect to certain state variables. It is shown that stability can be concluded via a sign-indefinite function, which explicitly takes the systems' periodicity into account. The presented approach leverages some of the difficulties encountered in the analysis of periodic systems via positive definite Lyapunov functions proposed in Angeli and Efimov (2013, 2015). The new result is established based on the framework of cell structure introduced in Leonov (1974) and illustrated via the global analysis of a nonlinear pendulum with a constant persistent input.

Keywords: Input-to-state stability, Periodic system, Multistable system

1. INTRODUCTION

Stability of dynamical systems is one of the fundamental problems studied in control systems theory Bhatia and Szegö (1970); Chetaev (1961); Gelig et al. (1978); Hahn (1967); Khalil (1992); Lakshmikantham and Liu (1993); Lyapunov (1992) and related domains, such as mechanics, electric circuits, power systems, systems biology, *etc.* In a general (nonlinear) setting, the main approach employed for stability analysis is based on Lyapunov theory Lyapunov (1992). A key advantage of a Lyapunov-based stability analysis is that boundedness and convergence properties of the solutions can be assessed without explicit computation of these solutions. Instead, it suffices to verify some inequalities for the Lyapunov function and its time derivative, which is derived with respect to the system equations. More precisely, the existence of a continuously differentiable (or at least Lipschitz continuous) Lyapunov function, which is positive definite with respect to an equilibrium (or an invariant set) and the time derivative of which is negative definite along the solutions of the system under investigation, is equivalent to stability of the equilibrium (or the set) of that dynamical system. Similarly, instability of an equilibrium can be studied using the Chetaev function approach Chetaev (1961); Efimov et al. (2014). A Chetaev function may be sign indefinite with a negative definite derivative. There are several extensions of Lyapunov theory, including input-to-state stability (ISS) and related notions Sontag (1995); Dashkovskiy et al. (2011) and uniform stability Lin et al. (1996), all of which allow to account for robustness in the presence of external inputs.

Classical stability theory is mainly concerned with the analysis of a single equilibrium. However, in numerous applications, such as biological or power systems, there exist several equilibria or invariant sets. Hence, the rigorous analysis of such systems with several disjoint invariant sets represents an important special case of stability theory, which requires suitable methods Angeli et al. (2004); Nitecki and Shub (1975); Gelig et al. (1978); Rantzer (2001); Angeli and Sontag (2004); Efimov and Fradkov (2009); Efimov (2012). For this case the stability notions have to be significantly modified and relaxed as, in particular, it has been done in Efimov (2012) and further in Angeli and Efimov (2013, 2015) for the ISS case. See also Angeli (2004); Angeli and Praly (2011); Chaves et al. (2008) for other results on robust stability analysis of multistable systems. The main result of Angeli and Efimov (2015) provides necessary and sufficient conditions under which

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a system is stable with respect to multiple invariant solutions, which belong to a decomposable set (see Definition 3 below). Then, stability of the system with respect to this decomposable set is equivalent to 1) the existence of a nonnegative (taking zero value on some of that sets) Lyapunov function, which is continuously differentiable on the manifold where the system dynamics evolves and 2) the time derivative of this Lyapunov function is negative definite along the solutions of the system and only vanishes at elements of the decomposable invariant set.

The present paper extends the results of Angeli and Efimov (2013, 2015) by relaxing the requirements on differentiability and positive definiteness of the Lyapunov function. For this purpose, we consider a special class of systems with periodic right-hand sides with respect to a part of the state vector. This kind of dynamics is ubiquitous in the area of power systems, which has attracted the attention of researchers in the last few years Ortega et al. (2005); Schiffer et al. (2014, 2015a); Efimov et al. (2015); Schiffer et al. (2015b); Efimov et al. (2016). To establish the result, we use the framework of *cell structure* proposed in Leonov (1974) (and later in Noldus (1977)) and developed in Gelig et al. (1978); Yakubovich et al. (2004) for autonomous systems. As in Angeli and Efimov (2013, 2015), under the aforementioned relaxed assumptions, this permits to derive necessary and sufficient conditions for ISS. The derived framework is tested by applying it to a nonlinear pendulum with constant permanent input.

The outline of this paper is as follows. Preliminaries and the theories from Angeli and Efimov (2015) and Gelig et al. (1978); Yakubovich et al. (2004) are given in Section 2. The problem statement is given in Section 3 with the main results in Section 4. The efficiency of the presented robust stability conditions is illustrated by means of the example of a nonlinear pendulum in Section 5.

2. PRELIMINARIES

For an n-dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold M without a boundary, let the map $f(x,d) : M \times \mathbb{R}^m \to T_x M$ be of class $\mathcal{C}^1$, and consider a nonlinear system of the following form:

$$\dot{x}(t) = f(x(t), d(t)), \tag{1}$$

where the state $x(t) \in M$ and $d(t) \in \mathbb{R}^m$ (the input $d(\cdot)$ is a locally essentially bounded and measurable signal) for $t \geq 0$. We denote by $X(t, x_0; d)$ the uniquely defined solution of (1) at time t fulfilling $X(0, x_0; d) = x_0$. Together with (1) we will analyze its unperturbed version:

$$\dot{x}(t) = f(x(t), 0). \tag{2}$$

A set $S \subset M$ is invariant for the unperturbed system (2) if $X(t, x; 0) \in S$ for all $t \in \mathbb{R}$ and for all $x \in S$. Define the distance from a point $x \in M$ to the set $S \subset M$ as $|x|_S = \min_{a \in S} \delta(x, a)$, where the symbol $\delta(x_1, x_2)$ denotes the Riemannian distance between x_1 and x_2 in M, $|x| = |x|_{\{0\}}$ for $x \in M$ (0 is a point selected on M) or a usual Euclidean norm of a vector $x \in \mathbb{R}^n$. For a signal $d : \mathbb{R} \to \mathbb{R}^m$ the essential supremum norm is defined as $\|d\|_\infty = \operatorname{ess\,sup}_{t \geq 0} |d(t)|$.

A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$ and $\lim_{t \to +\infty} \beta(s, t) = 0$ for each fixed $s \in \mathbb{R}_+$.

2.1 Decomposable sets

Let $\Lambda \subset M$ be a compact invariant set for (2).

Definition 1. Nitecki and Shub (1975) A decomposition of Λ is a finite and disjoint family of compact invariant sets $\Lambda_1, \ldots, \Lambda_k$ such that

$$\Lambda = \bigcup_{i=1}^{k} \Lambda_i.$$

For an invariant set Λ, its attracting and repulsing subsets are defined as follows:

$$\mathfrak{A}(\Lambda) = \{x \in M : |X(t, x; 0)|_\Lambda \to 0 \text{ as } t \to +\infty\},$$
$$\mathfrak{R}(\Lambda) = \{x \in M : |X(t, x; 0)|_\Lambda \to 0 \text{ as } t \to -\infty\}.$$

Define a relation on $\mathcal{W} \subset M$ and $\mathcal{D} \subset M$ by $\mathcal{W} \prec \mathcal{D}$ if $\mathfrak{A}(\mathcal{W}) \cap \mathfrak{R}(\mathcal{D}) \neq \oslash$.

Definition 2. Nitecki and Shub (1975) Let $\Lambda_1, \ldots, \Lambda_k$ be a decomposition of Λ, then

1. An r-cycle ($r \geq 2$) is an ordered r-tuple of distinct indices $i_1, \ldots, i_r$ such that $\Lambda_{i_1} \prec \ldots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.

2. A 1-cycle is an index i such that $[\mathfrak{R}(\Lambda_i) \cap \mathfrak{A}(\Lambda_i)] - \Lambda_i \neq \oslash$.

3. A filtration ordering is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \Rightarrow i \leq j$.

As we can conclude from Definition 2, existence of an r-cycle with $r \geq 2$ is equivalent to existence of a heteroclinic cycle for (2) Guckenheimer and Holmes (1988). Furthermore, existence of a 1-cycle implies existence of a homoclinic cycle for (2) Guckenheimer and Holmes (1988).

Definition 3. The set $\mathcal{W}$ is called decomposable if it admits a finite decomposition without cycles, $\mathcal{W} = \bigcup_{i=1}^{k} \mathcal{W}_i$, for some non-empty disjoint compact sets $\mathcal{W}_i$, which form a filtration ordering of $\mathcal{W}$ (as in definitions 1 and 2).

2.2 Robustness notions

The following robustness notions for systems represented by (1) have been introduced in Angeli and Efimov (2013, 2015) (see also Dashkovskiy et al. (2011) for a survey on the ISS framework).

Definition 4. We say that (1) has the practical asymptotic gain (pAG) property with respect to $\mathcal{W}$ if there exist $\eta \in \mathcal{K}_\infty$ and a non-negative real q such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\limsup_{t \to +\infty} |X(t, x; d)|_{\mathcal{W}} \leq \eta(\|d\|_\infty) + q.$$

If $q = 0$, then we say that the asymptotic gain (AG) property holds.

Definition 5. We say that the system (1) has the limit property (LIM) with respect to $\mathcal{W}$ if there exists $\mu \in \mathcal{K}_\infty$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\inf_{t\geq 0} |X(t,x;d)|_{\mathcal{W}} \leq \mu(\|d\|_\infty).$$

Definition 6. We say that the system (1) has the practical global stability (pGS) property with respect to $\mathcal{W}$ if there exist $\beta \in \mathcal{K}_\infty$ and $q \geq 0$ such that for all $x \in M$ and measurable essentially bounded inputs $d(\cdot)$ the following holds for all $t \geq 0$:

$$|X(t,x;d)|_{\mathcal{W}} \leq q + \beta(\max\{|x|_{\mathcal{W}}, \|d\|_\infty\}).$$

It has been shown in Angeli and Efimov (2013, 2015) that to characterize pAG property in terms of Lyapunov functions the following notion is appropriate.

Definition 7. We say that a $\mathcal{C}^1$ function $V : M \to \mathbb{R}_+$ is a practical ISS Lyapunov function for (1) if there exists $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ and γ, and scalars $q \geq 0$ and $c \geq 0$ such that

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{W}} + c),$$

the function V is constant on each $\mathcal{W}_i$ and the following dissipation holds:

$$DV(x)f(x,d) \leq -\alpha_3(|x|_{\mathcal{W}}) + \gamma(|d|) + q.$$

If the latter inequality holds for $q = 0$, then V is said to be an ISS Lyapunov function.

Note that existence of α_2 and c follows (without any additional assumptions) by standard continuity arguments.

The main result of Angeli and Efimov (2013, 2015) relating these robust stability properties is stated below, it extends the results of Sontag and Wang (1995, 1996) obtained for connected sets.

Theorem 8. Consider a nonlinear system as in (1) and let a compact invariant set containing all α- and ω-limit sets of (2) $\mathcal{W}$ be decomposable (in the sense of Definition 3). Then the following facts are equivalent.

1. The system admits an ISS Lyapunov function;
2. The system enjoys the AG property;
3. The system admits a practical ISS Lyapunov function;
4. The system enjoys the pAG property;
5. The system enjoys the LIM property and the pGS.

Definition 9. Angeli and Efimov (2015) Suppose that a nonlinear system as in (1) satisfies the assumptions and the list of equivalent properties of Theorem 8. Then this system is called ISS with respect to the set $\mathcal{W}$.

2.3 Boundedness of solutions of periodic systems

As outlined in Section 1, the present paper is dedicated to the stability analysis of periodic systems Gelig et al. (1978); Yakubovich et al. (2004). More precisely, for the system (1) there exists $\xi \in \mathbb{R}^n$, $\xi \neq 0$, such that

$$f(x,0) = f(x+\xi,0)$$

for all $x \in \mathbb{R}^n$. Roughly speaking, in such a case there exists a coordinate transformation such that $M = \mathbb{R}^k \times \mathbb{S}^q$, where $n = k + q$ and $\mathbb{S}$ is the unit sphere.

Next, we recall a sufficient criterion derived in Leonov (1974); Gelig et al. (1978); Yakubovich et al. (2004), which allows to establish *boundedness* of solutions of periodic systems. To this end consider a special case of (2):

$$f(x,0) = Px + b\varphi(c^Tx)$$

with $M = \mathbb{R}^n$, where $P \in \mathbb{R}^{n\times n}$ is a singular matrix, $c, b \in \mathbb{R}^n$, $\varphi : \mathbb{R} \rightrightarrows \mathbb{R}$ is a Δ-periodical set-valued function, which is upper semicontinuous, with a nonempty, convex and closed set of values for any value of its argument. We note that a time-varying version of φ has been considered in Gelig et al. (1978); Yakubovich et al. (2004), but we restrict ourselves to autonomous version of φ. Then under these restrictions and for any initial condition $x_0 \in \mathbb{R}^n$ the system (2) has a solution $X(t,x_0;0)$. Assume also that for all $\sigma \in \mathbb{R} \setminus \{0\}$ and all $\phi \in \varphi(\sigma)$

$$\mu_1 \leq \frac{\phi}{\sigma} \leq \mu_2;\ \mu_1^{-1}\mu_2^{-1}\varphi(0) = 0$$

for some $\mu_1 \in \mathbb{R} \cup \{-\infty\}$ and $\mu_2 \in \mathbb{R} \cup \{+\infty\}$. The periodicity of φ implies that either $\mu_1 < 0$, $\mu_2 > 0$ or $\mu_1 = \mu_2 = 0$, and the latter case is excluded from consideration due to its triviality.

Theorem 10. Leonov (1974); Gelig et al. (1978); Yakubovich et al. (2004) Assume that there exists $\lambda > 0$ such that:

1) the matrix $P + \lambda I_n$, where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix, has $n-1$ eigenvalues with negative real parts;

2) for all $\omega \in \mathbb{R}$

$$\mu_1^{-1}\mu_2^{-1} + (\mu_1^{-1} + \mu_2^{-1})\mathrm{Re}\chi(i\omega - \lambda) + |\chi(i\omega - \lambda)|^2 \leq 0,$$

where $\chi(s) = c^T(P - sI_n)^{-1}b$.

Then, for any initial condition $x_0 \in \mathbb{R}^n$ the solution $X(t,x_0;0)$ of (2) is bounded for $t \in [0,+\infty)$.

To prove this theorem (see Theorem 4.3.1 in Gelig et al. (1978), or Theorem 4.7 in Yakubovich et al. (2004)) note that under introduced conditions there is $H = H^T \in \mathbb{R}^{n\times n}$ (it has one negative and $n-1$ positive eigenvalues) such that for $V_0(x) = x^THx$ we have $dV_0(x(t))/dt \leq -2\lambda V_0(x(t))$ for all $t \in [0,+\infty)$, which implies that the set $\Omega_0 = \{x \in \mathbb{R}^n : V_0(x) \leq 0\}$ is invariant for (2), *i.e.* $X(t,x_0;0) \in \Omega_0$ for all $t \in [0,+\infty)$ provided that $x_0 \in \Omega_0$. Next, introducing the functions $V_j(x) = V_0(x - j\delta)$ and sets $\Omega_j = \{x \in \mathbb{R}^n : V_j(x) < 0\}$, where j is any integer and the vector $\delta \in \mathbb{R}^n$ satisfies the conditions $\delta \neq 0$, $P\delta = 0$ and $c^T\delta = \Delta$, by periodicity of f in (2) we obtain that $dV_j(x(t))/dt \leq -2\lambda V_j(x(t))$ for all $t \in [0,+\infty)$, then the set Ω_j is invariant for (2). Finally, it is shown that for any $x_0 \in \mathbb{R}^n$ there is an index j_0 such that $x_0 \in \Gamma_{j_0}$, where $\Gamma_j = \Omega_j \cap \Omega_{-j} \cap \{x \in \mathbb{R}^n : |h^Tx| \leq j|h^T\delta|\}$ with $h \in \mathbb{R}^n$ being the eigenvector of the matrix H corresponding to the negative eigenvalue. As it has been shown above $X(t,x_0;0) \in \Gamma_{j_0}$ for all $t \in [0,+\infty)$ (since it is true for $\Omega_{j_0} \cap \Omega_{-j_0}$). In addition the set Γ_{j_0} is bounded, which was necessary to prove. In other words, an important observation of Leonov (1974); Gelig et al. (1978); Yakubovich et al. (2004) is that any intersection of the sets Ω_j for all integers j forms a kind of cell cover of $\mathbb{R}^n$, where each cell is bounded and invariant.

3. PROBLEM STATEMENT

The main contribution of the present work is the derivation of necessary and sufficient conditions under which a periodic system possesses the ISS properties given in Definition 9. This is achieved by combining the cell structure approach presented in the proof of Theorem 10 (and firstly introduced in Leonov (1974)) with the ISS approach for multistable systems of Angeli and Efimov (2013, 2015). The fundamental difference between the theories given in subsections 2.2 and 2.3 is that the former makes analysis on a manifold M, while the latter one considers a multistable system in $\mathbb{R}^n$. To this end, let $M = \mathbb{R}^k \times \mathbb{S}^q$ with $n = k + q$ and denote $x = (z, \theta) \in M$ with $z \in \mathbb{R}^k$ and $\theta \in \mathbb{S}^q$, then by embedding (2) in $\mathbb{R}^n$ and due to continuity of f we obtain that for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$

$$f(\tilde{x}, 0) = f(\tilde{x} + \xi, 0),\ \xi = [\underbrace{0, \dots, 0}_{k}, \underbrace{2\pi, \dots 2\pi}_{q}] \in \mathbb{R}^n.$$

In this case for any $\tilde{x}_0 \in \mathbb{R}^n$ there is a unique and, at least, locally in time defined solution of the system (1) $\tilde{X}(t, \tilde{x}_0; d) \in \mathbb{R}^n$. Denote

$$\mathcal{P} : \mathbb{R}^n \to M$$

as the projection from $\mathbb{R}^n$ to M (that is just a modulus of the last q coordinates over 2π). Obviously, for any $\tilde{x}_0 \in \mathbb{R}^n$ the solution $\tilde{X}(t, \tilde{x}_0; d) \in \mathbb{R}^n$ of (1) can be projected to the solution $X(t, x_0; d) \in M$ with $x_0 = \mathcal{P}(\tilde{x}_0) \in M$, then both solutions are defined on the same interval of time and $X(t, x_0; d) = \mathcal{P}(\tilde{X}(t, \tilde{x}_0; d))$ for all such instants of time. Similarly, the set $\mathcal{W} \subset M$, containing all α- and ω-limit sets of (2), can be extended to whole $\mathbb{R}^n$ using periodicity of the last q variables, which we will denote as $\tilde{\mathcal{W}}$ (but the set $\tilde{\mathcal{W}}$ becomes unbounded in $\mathbb{R}^n$, in a common case), then $|\tilde{x}|_{\tilde{\mathcal{W}}} = \inf_{y \in \tilde{\mathcal{W}}} |\tilde{x} - y|$ is a distance to that set for $\tilde{x} \in \mathbb{R}^n$.

The ISS Lyapunov function introduced in Definition 7 should be positive definite with respect to distance to the set $\mathcal{W}$, while the functions proposed in Leonov (1974) for analysis of boundedness of trajectories of periodic system (2) are sign indefinite. Usually sign indefinite functions with a sign definite derivative are used for establishment of instability in (2), *e.g.* Chetaev functions Chetaev (1961); Efimov et al. (2014). However, for the periodic systems this drawback is overcome in Leonov (1974) using the system period. Clearly, such a relaxation of definiteness of Lyapunov function can simplify a lot applications of the method, then inspired by Leonov (1974) we will define the following characterization of ISS property with respect to the set $\mathcal{W}$ for a periodic system:

Definition 11. We say that a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ is a practical ISS Leonov function for (1) with $M = \mathbb{R}^k \times \mathbb{S}^q$ if there exist functions $\alpha_1, \alpha_2, \sigma, \gamma \in \mathcal{K}_\infty$, a continuous function $\lambda : \mathbb{R} \to \mathbb{R}$, $\lambda \in \mathcal{K}_\infty$ for nonnegative arguments, and scalars $r \geq 0$, $g \geq 0$ such that for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$

$$\alpha_1(|\tilde{x}|_{\tilde{\mathcal{W}}}) - \sigma(|\tilde{\theta}|) \leq V(\tilde{x}) \leq \alpha_2(|\tilde{x}|_{\tilde{\mathcal{W}}} + g), \qquad (3)$$

and the following dissipation holds:

$$DV(\tilde{x}) f(\tilde{x}, d) \leq -\lambda(V(\tilde{x})) + \gamma(|d|) + r. \qquad (4)$$

If the latter inequality holds for $r = 0$, then V is said to be an ISS Leonov function.

Let us stress that an ISS Leonov function V can be continuously differentiable on $\mathbb{R}^n$, but discontinuous on M, while an ISS-Lyapunov function should be continuously differentiable on M (*i.e.* in this case V should be 2π-periodic in θ), which is another relaxation in Definition 11 compared to Definition 7. Therefore, any ISS-Lyapunov function is a practical ISS Leonov function for a periodic system (1) since for any $\tilde{x} \in \mathbb{R}^n$ and any $\sigma \in \mathcal{K}_\infty$:

$$\alpha_1(|\tilde{x}|_{\mathcal{W}}) - \sigma(|\tilde{\theta}|) \leq \alpha_1(|\tilde{x}|_{\mathcal{W}}),$$
$$-\alpha_3(|\tilde{x}|_{\mathcal{W}}) \leq -\alpha_3(0.5[|\tilde{x}|_{\mathcal{W}} + c]) + \alpha_3(c)$$
$$\leq -\alpha_3(0.5\alpha_2^{-1}(V(\tilde{x}))) + \alpha_3(c).$$

Remark 12. If $0 \in \mathcal{W}$, then without loosing generality the property (3) can be replaced in Definition 11 by the following one:

$$\alpha_1(|\tilde{z}|) - \sigma(|\tilde{\theta}|) \leq V(\tilde{x}) \leq \alpha_2(|\tilde{x}|_{\tilde{\mathcal{W}}} + g). \qquad (5)$$

In the remainder of this work, it is shown that the existence of ISS Leonov function is an equivalent characterization of ISS property from Definition 9 for a periodic system (1).

4. MAIN RESULT

If $V : M \to \mathbb{R}$ is a continuously differentiable function admitting relations (3) for all $x \in M$ and some $\alpha_1, \alpha_2, \sigma \in \mathcal{K}_\infty$, then by adding a constant $w > 0$ the new function $V(x) + w$ can be made positive definite. Therefore, the definition of V as a function from $\mathbb{R}^n$ to $\mathbb{R}$ is crucial in Definition 11.

All proofs of the following results are omitted due to space limitations.

Lemma 13. Let $M = \mathbb{R}^k \times \mathbb{S}^q$ with $n = k + q$, and $\mathcal{W} \subset M$ be a compact invariant set. Then existence of a practical ISS Leonov function for (1) implies pGS and pAG properties with respect to $\mathcal{W}$.

Theorem 14. Let $M = \mathbb{R}^k \times \mathbb{S}^q$ with $n = k + q$, and a compact invariant set containing all α- and ω-limit sets of (2) $\mathcal{W} \subset M$ be decomposable (in the sense of Definition 3). Then, for (1) the following properties are equivalent:

(a) ISS with respect to the set $\mathcal{W}$;

(b) there is a practical ISS Leonov function.

Note that from this result, an ISS Leonov function is only sufficient for the ISS property of (1) in general:

Corollary 15. Let $M = \mathbb{R}^k \times \mathbb{S}^q$ with $n = k + q$, and a compact invariant set containing all α- and ω-limit sets of (2) $\mathcal{W}$ be decomposable (in the sense of Definition 3). Then for (1) existence of an ISS Leonov function implies the ISS property with respect to the set $\mathcal{W}$.

The practical interest of the proposed theory is illustrated via a benchmark example taken from Forni and Sepulchre (2014) in the next section.

5. APPLICATION TO A NONLINEAR PENDULUM

Consider a nonlinear pendulum with a biased external input:

$$\dot{\theta}(t) = z(t), \qquad (6)$$
$$\dot{z}(t) = -\kappa z(t) - \omega^2 \sin(\theta(t)) + c + d(t),$$

where $\theta(t) \in \mathbb{S}$ and $z(t) \in \mathbb{R}$ are angular position and angular velocity of the pendulum, $x = (z, \theta) \in M = \mathbb{R} \times \mathbb{S}$, $\kappa > 0$ and $\omega > 0$ are two parameters, $c \in \mathbb{R}$ is the input bias, $d(t) \in \mathbb{R}$ is an external disturbance (a locally essentially bounded and measurable signal).

The unperturbed system (6), for $c + d(t) \equiv 0$, has two equilibria $[0, 0]$ and $[\pi, 0]$ (the former is attractive and the latter one is a saddle-point). Thus, in this case $\mathcal{W} = \{[0, 0] \cup [\pi, 0]\}$ is a compact set containing all α- and ω-limit sets of (6). In addition, it is straightforward to check that $\mathcal{W}$ is decomposable in the sense of Definition 3. An ISS property of (6) has been shown in Angeli and Efimov (2013, 2015) and an ISS Lyapunov function for (6) has been proposed in Efimov et al. (2015). Using that result, for the case $|c| < \omega^2$, the global convergence to one of the two equilibria $[\mathrm{asin}(c\omega^{-2}), 0]$ or $[\pi - \mathrm{asin}(c\omega^{-2}), 0]$ has been proven in Efimov et al. (2016) under some restrictions on values of parameters c, κ, ω and using an additional discontinuous Lyapunov function for a local analysis.

In this work we will show the ISS property of (6) under less restrictive conditions than in Efimov et al. (2016) and using the ISS Leonov function framework proposed above. For this purpose, assume that $|c| < \omega^2$ and consider

$$V(x) = 0.5z^2 + \omega^2 w(\theta - \theta_0),$$
$$w(s) = \cos(\theta_0) - \cos(s + \theta_0) - \sin(\theta_0)s - u\cos(\theta_0)s^2,$$

where $\theta_0 = \mathrm{asin}(c\omega^{-2})$ and $u \in \mathbb{R}$ is a parameter to be defined later. Note that w is not periodic in θ, thus V cannot be an ISS-Lyapunov function, but it can be considered as an ISS Leonov function candidate. Straightforward calculations yield:

$$w'(s) = \sin(s + \theta_0) - \sin(\theta_0) - 2u\cos(\theta_0)s,$$
$$w''(s) = \cos(s + \theta_0) - 2u\cos(\theta_0).$$

Since $\cos(\theta_0) = \sqrt{1 - c^2\omega^{-4}} > 0$, then $w''(0) < 0$ for $u > 0.5$ and there exist $u^* > 0.5$ such that $w'(s) = 0$ only for $s = 0$ with $u \geq u^*$ (the equation $w''(s) = 0$ has no solution for a sufficiently high u and $w'(s)$ is strictly decreasing in such a case), which together with the property $w''(0) < 0$ implies that $w(s) < 0$ for all $s \neq 0$ and $w(0) = 0$. Therefore, for $u > u^*$there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that

$$-\epsilon_1 s^2 \leq w(s) \leq -\epsilon_2 s^2,$$

then for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^2$:

$$0.5\tilde{z}^2 - \epsilon_1\omega^2(\tilde{\theta} - \theta_0)^2 \leq V(\tilde{x}) \leq 0.5\tilde{z}^2 - \epsilon_2\omega^2(\tilde{\theta} - \theta_0)^2$$

and the relations (3) are satisfied. Let us check (4):

$$\begin{aligned}\dot{V} &= \tilde{z}d - \kappa\tilde{z}^2 - 2u\omega^2\cos(\theta_0)(\tilde{\theta} - \theta_0)\tilde{z} \\ &\leq \frac{d^2}{2\kappa} - \frac{\kappa}{4}\tilde{z}^2 + \frac{4}{\kappa}u^2\omega^4\cos^2(\theta_0)(\tilde{\theta} - \theta_0)^2 \\ &= \frac{d^2}{2\kappa} - \frac{\kappa}{2}\left(\frac{\tilde{z}^2}{2} - \epsilon_2\omega^2(\tilde{\theta} - \theta_0)^2\right) \\ &\quad + \left(\frac{4}{\kappa}u^2\omega^4\cos^2(\theta_0) - \frac{\kappa}{2}\epsilon_2\omega^2\right)(\tilde{\theta} - \theta_0)^2.\end{aligned}$$

Assume that $\frac{\omega^2\cos^2(\theta_0)}{\kappa^2} \leq \frac{\epsilon_2}{8u^2}$ or equivalently

$$\frac{\omega^4 - c^2}{\omega^2\kappa^2} \leq \frac{\epsilon_2}{8u^2}, \tag{7}$$

then the last term in the estimate for $\dot{V}$ becomes non-positive, and finally we obtain:

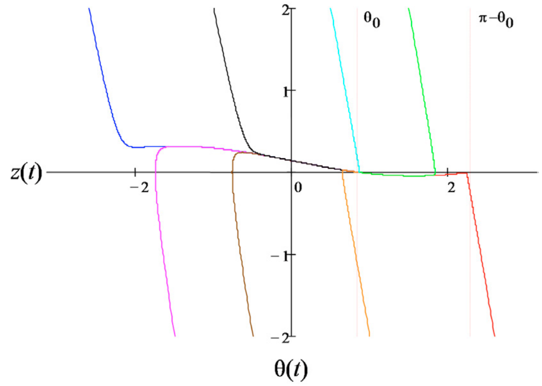

Figure 1. Simulation results for the system (6) with $d(t) = 0$ and for several arbitrarily chosen initial conditions

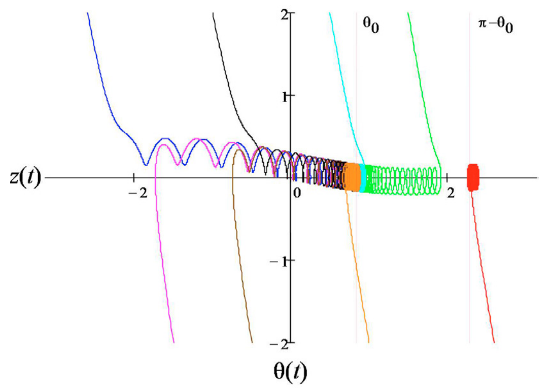

Figure 2. Simulation results for the system (6) with $d(t) = 1.1\sin(4t)$ and for several arbitrarily chosen initial conditions

$$\dot{V} \leq \frac{d^2}{2\kappa} - \frac{\kappa}{2}\left(\frac{\tilde{z}^2}{2} - \epsilon_2\omega^2(\tilde{\theta} - \theta_0)^2\right) \leq \frac{d^2}{2\kappa} - \frac{\kappa}{2}V$$

and V is an ISS Leonov function for (6). Taking $d = 0$ it is easy to prove Efimov et al. (2016) that all solutions are bounded in that case and converge to one of the equilibria: $[\mathrm{asin}(c\omega^{-2}), 0]$ or $[\pi - \mathrm{asin}(c\omega^{-2}), 0]$. Then under the restriction on parameters (7) (ϵ_2 and u are also some functions of c, κ, ω), $\mathcal{W} = \left\{[\mathrm{asin}(c\omega^{-2}), 0] \cup [\pi - \mathrm{asin}(c\omega^{-2}), 0]\right\}$ is a compact set containing all α- and ω-limit sets of (6) for $d = 0$, and it is is decomposable in the sense of Definition 3. Finally, by Theorem 14 the system (6) is ISS with respect to $\mathcal{W}$ under the restriction (7).

Example 1. Select $c = 0.75$ and $\omega = 1$, then $u = \frac{3}{5}$, $\epsilon_1 = 1$ and $\epsilon_2 = \frac{1}{24}$ is an acceptable choice, and for any κ such that the restriction (7) is satisfied, that is

$$\kappa \geq \frac{6}{5}\sqrt{21} \simeq 5.499,$$

the system (6) is ISS with respect to $\mathcal{W} = \{[0.848, 0] \cup [2.294, 0]\}$. Examples of the system trajectories with $\kappa = 5.5$ and $d = 0$ are given in Fig. 1, and for $d(t) = 1.1\sin(4t)$ in Fig. 2 (the unstable equilibrium captures one of the trajectories). Clearly, the simulations confirm the conclusions of the proposed theory.

6. CONCLUSIONS

We have derived necessary and sufficient conditions of ISS property for multistable periodic systems, *i.e.*, systems whose dynamics is periodic with respect to a part of the state variables. To prove this result and by building upon pioneering ideas in Leonov (1974), we have introduced the concept of an ISS Leonov function. Such a function can be sign indefinite and not continuously differentiable on the manifold where the system dynamics evolves. These represent significant relaxations compared to the usual requirements on a standard ISS Lyapunov function Angeli and Efimov (2013, 2015). The proposed approach is illustrated by providing a global analysis of a nonlinear pendulum with constant input. We expect the derived methodology to be applicable to many challenging engineering problems and plan to investigate this in future works.

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